Working with Imaginary Numbers

We use the notation i to help us represent imaginary numbers

$i = \sqrt{-1}$

$i^2 = -1$

Examples for when we might use this would be when we want to simplify square roots that are negative, or find all of the answers to a polynomial equation.

For example if we want to simplify $\sqrt{-5}$, we can rewrite that as $\sqrt{-1} * \sqrt{5}$. Since $\sqrt{-1} = i$, we would have $i\sqrt{5}$ as our simplified answer.

Here are some more examples of using i to help you simplify:

$\sqrt{-100} = \sqrt{-1} * \sqrt{100} = i * 10 = 10i$

$\sqrt{-28} = \sqrt{-1} * \sqrt{28} = i\sqrt{28} = i * \sqrt{4} * \sqrt{7} = 2i\sqrt{7}$

$\sqrt{-3} * \sqrt{-3} = i\sqrt{3} * i\sqrt{3} = i^2\sqrt{9} = 3i^2 = -3$

Simplifying expressions of the form $\boldsymbol{i^x}$

1. If x is an odd number, split the term up as $i^{x-1} * i$. This makes it easier to solve, as x-1 will be an even number.
2. If x is initially an even number, proceed to the next step.
3. Take the term with the even exponent, and rewrite it using i^2 and raised to another power; half of the even exponent
    a. For example, if it is i^{10}, it would become $(i^2)^5$, as 5 is half of 10 (Also, 2∗5=10)
4. Replace the i^2 term with -1
5. Perform the operation for the -1 term
6. If needed, multiply the answer with the remaining i

Examples:

$i^{29} = i^{28} * i = (i^2)^{14} * i = (-1)^{14} * i = 1 * i = i$

$i^{54} = (i^2)^{27} = (-1)^{27} = -1$

$i^{43} = i^{42} * i = (i^2)^{21} * i = (-1)^{21} * i = -1 * i = -i$

$i^{88} = (i^2)^{44} = (-1)^{44} = 1$

Using imaginary numbers in solving quadratic equations

The general form of a solution to a quadratic equation with an imaginary number as part of the solution is $a \pm bi$, where a and b are both real numbers. We will see this through the following examples.

Solve the equation: $x^2 + 2x + 5$

Using the quadratic equation, we would have:

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm i\sqrt{16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

So the solutions are $-1 + 2i$ and $-1 - 2i$

Solve the equation: $3x^2 - 6x + 14$

Using the quadratic equation, we would have:

$$\frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(14)}}{2(3)} = \frac{6 \pm \sqrt{36 - 168}}{6}$$

$$= \frac{6 \pm \sqrt{-132}}{6}$$

$$= \frac{6 \pm i\sqrt{132}}{6}$$

$$= \frac{6 \pm i\sqrt{4}\sqrt{33}}{6}$$

$$= \frac{6 \pm 2i\sqrt{33}}{6}$$

$$= \frac{3 \pm i\sqrt{33}}{3}$$

So the solutions we have right now are $\frac{3}{3} + \frac{i\sqrt{33}}{3}$ and $\frac{3}{3} - \frac{i\sqrt{33}}{3}$, which can be simplified to:

$$1 + \frac{i\sqrt{33}}{3} \text{ and } 1 - \frac{i\sqrt{33}}{3}$$

Rationalizing Denominators with Imaginary Numbers

Say we had the fraction $\frac{4-i}{2+3i}$ and we want to rationalize the denominator. Here are the steps (This process is very similar to rationalizing denominators with square roots):

1. Multiply both the numerator and denominator by the conjugate of the denominator.
   a. In our example, this would mean multiplying by $2 - 3i$ on both the numerator and the denominator
   b. This would look like $\frac{4-i}{2+3i} * \frac{2-3i}{2-3i}$
2. FOIL (or distribute if necessary) the numerator and denominator, and simplify each part.
   a. In our example, we would have $\frac{8-12i-2i+3i^2}{4-6i+6i-9i^2}$. This simplifies to $\frac{8-14i+3i^2}{4-9i^2}$

3. Substitute -1 in for any i^2 terms and simplify even more.
    a. For our example, substituting -1 in for i^2 would leave us with $\frac{8-14i+3(-1)}{4-9(-1)}$. This simplifies to $\frac{8-14i-3}{4+9}$ which simplifies even further to $\frac{5-14i}{13}$
4. Write your answer in the form $a+bi$. Simplify if necessary
    a. For our example, this would be $\frac{5}{13}-\frac{14i}{13}$, which is our final answer as neither fraction can be simplified.

Example: Simplify the expression $\frac{3+5i}{6-7i}$

The conjugate of the denominator is $6+7i$, so we use that to help us simplify.

$\frac{3+5i}{6-7i} * \frac{6+7i}{6+7i} = \frac{18+21i+30i+35i^2}{36+42i-42i-49i^2}$	*Multiply by the conjugate and FOIL
$= \frac{18+51i+35i^2}{36-49i^2}$	*Simplify any like terms
$= \frac{18+51i+35(-1)}{36-49(-1)}$	*Substitute -1 in for i^2
$= \frac{18+51i-35}{36+49}$	*Simplify by multiplying the -1 terms
$= \frac{-17+51i}{85}$	*Simplify any like terms
$= -\frac{17}{85}+\frac{51i}{85}$	*Write in the form $a+bi$
$= -\frac{1}{5}+\frac{3i}{5}$	*Simplify the fractions by dividing by 17 on both